TOTAL DIFFERENTIABILITY.

BY E. J. TOWNSEND.

Suppose we have given a single-valued function $z = f(x, y)$ of two real variables, defined for a region R given by the inequalities

$$a < x < b, \qquad c < y < d.$$

Thomae* seems to have been the first to point out that the mere existence of the partial derivatives $f_x'(x_0, y_0)$, $f_y'(x_0, y_0)$ is not sufficient for the total differentiability of the given function at the point (x_0, y_0).

More recently several writers† have formulated a more precise definition of a total differentiable. These definitions are, however, equivalent and may be stated as follows:

For convenience, denote by $\Delta(x, y)$ the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between any given point (x_0, y_0) of R and any other such point $(x_0 + \Delta x, y_0 + \Delta y)$. The function $f(x, y)$ is then said to be totally differentiable at (x_0, y_0) if there exist two constants A, B such that

$$(1) \qquad \underset{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}}{\mathrm{L}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - A\Delta x - B\Delta y}{\Delta(x, y)} = 0.$$

As Fréchet points out, one might make use of $|\Delta x| + |\Delta y|$ instead of $\Delta(x, y)$. It follows from this definition that if the given function is totally differentiable, the partial derivatives $f_x'(x_0, y_0)$, $f_y'(x_0, y_0)$ both exist and are finite, A being nothing else than $f_x'(x_0, y_0)$ and B, $f_y'(x_0, y_0)$.

Such a definition meets adequately the needs of analysis and has been adopted in one form or another in recent texts.‡

We shall examine some of the consequences of such a definition, particularly the conditions that must be placed upon the partial derivatives f_x', f_y' in order that the given function may be said to be totally differentiable. As an aid to the geometrical interpretation of the results of such a study, it may be pointed out that a necessary and sufficient condition for total differentiability for $x = x_0$, $y = y_0$ is that the surface $z = f(x, y)$ shall

* See Theorie der Bestimmten Integrale (1875), p. 36.

† See Stotz, Differential- und Integral-rechnung (1893), p. 131; W. H. Young, Fundamental Theorems of Differential Calculus (1910), p. 21; Fréchet, Nouvelles Annalles de Math. (1912), vol. 71, p. 389.

‡ See Pierpont, Theory of Functions of Real Variables, vol. I, p. 269; Kowalewski, Komplexen Veränderlichen, p. 186; de la Vallée Poussin, Cours d'Analyse Infinitesimale, 3d ed., p. 140.

have a tangent plane at the point (x_0, y_0, z_0), which is not parallel to the z-axis.*

Continuity of $f(x, y)$ as regards the two variables taken together is a consequence of total differentiability, but as with functions of a single variable a given function may be continuous throughout a given region but not be totally differentiable at any point of the region. Total differentiability depends upon the existence of the partial derivatives f_x', f_y', and the character of their continuity.

If f_x', f_y' both exist and one is continuous in x and y together, then it follows that $f(x, y)$ is totally differentiable.† It is well known that a function of two variables which is continuous in each variable throughout a region is also continuous in both together at a set of points everywhere dense in that region. It follows then that if f_x' and f_y' exist and one is continuous in x and in y, then $f(x, y)$ is totally differentiable at a set of points everywhere dense. The question naturally arises whether under the foregoing conditions the given function is not totally differentiable at every point. This is not the case, however, as the following illustration shows.

Ex. 1. Given the function

$$f(x, y) = \frac{x^2y}{x^4 + y^2},$$

where $0 < n \leq 1$, $0 < y \leq 1$, and $f(0, 0) = 0$. This function is not continuous in x and y together at the origin; for, we obtain different limiting values by taking different approaches to that point. As continuity of $f(x, y)$ in x and y together is a necessary condition for total differentiability, it follows that the given function is not totally differentiable for $x = 0$, $y = 0$.

For $x \neq 0$, $y \neq 0$, we have

$$f_x' = 2xy \cdot \frac{y^2 - x^4}{(y^2 + x^4)^2}.$$

For $(x = 0, y = 0)$, for $(x \neq 0, y = 0)$, and for $(x = 0, y \neq 0)$, we have $f_x' = 0$. For $(x \neq 0, y \neq 0)$, we have

$$f_y' = x^2 \frac{x^4 - y^2}{(x^4 + y^2)^2};$$

while for $(x = 0, y = 0)$, $(x = 0, y \neq 0)$, we have $f_y' = 0$. For $(x \neq 0, y = 0)$, we obtain $f_y' = 1/x^2$.

It follows then that the given function is not totally differentiable at

* Cf. Fréchet, Nouvelles Annales de Math., vol. 71, p. 436.

† Stolz, Differential- und Integral-rechnung, vol. I, p. 134.

the origin, yet in every neighborhood of that point including the origin itself, the two partial derivatives f_x', f_y' exist and one of them, namely f_x', is continuous in x alone and in y alone. The partial derivative f_y', is continuous everywhere with respect to y. It is also continuous with respect to x, except for $x = 0$, $y = 0$.

As already pointed out, if one of the partial derivatives, f_x', f_y', is continuous in x and in y, then the points at which $f(x, y)$ is totally differentiable forms a set everywhere dense in the given region. However, the points at which the given function $f(x, y)$ is not totally differentiable may also form a set of points everywhere dense in the same region, as the following example shows.

Ex. 2. Consider the function

$$F(s, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \varphi_n(s, t),$$

where $\varphi_n(s, t)$ is formed from the function considered in *Ex.* 1 by replacing x by $\sin^2 n\pi s$ and y by $\sin^2 n\pi t$; that is

$$\varphi_n(s, t) = \frac{\sin^4 n\pi s \cdot \sin^2 n\pi t}{\sin^8 n\pi s + \sin^4 n\pi t}.$$

The function given in *Ex.* 1 takes the value one-half for $y = x^2$ but for all other values of x and y in the given region it is less than one-half. The amount of the (x, y)-discontinuity at the origin is one-half. Since the sine is never greater than unity, we have

$$\sum_{n=1}^{\infty} \frac{1}{n!} \varphi_n(s, t) < \sum_{n=1}^{\infty} \frac{1}{n!}.$$

As the latter series converges, the series $\Sigma(1/n!)\varphi_n(s, t)$ converges uniformly as a function of the two variables (s, t) and hence as a function of either variable separately. As we shall see, the function $F(s, t)$ is not continuous in s and t together throughout the region of definition, because each term of the series is discontinuous at certain points in these two variables. However, $\varphi_n(s, t)$ is continuous in s alone and in t alone; and because of the uniform convergence of the above series, $F(s, t)$ is continuous in each variable separately. Consequently, the points at which it is continuous in both variables must form a set everywhere dense.

The points at which $F(s, t)$ has a discontinuity in s and t together also form a set everywhere dense; being those points where both s and t have rational values, as we shall now show. We have

$$F(s, t) = \frac{\sin^4 \pi s \cdot \sin^2 \pi t}{\sin^2 \pi s + \sin^4 \pi t} + \frac{1}{2!} \cdot \frac{\sin^4 2\pi s \cdot \sin^2 2\pi t}{\sin^8 2\pi s + \sin^4 2\pi t} + \cdots \tag{2}$$

$$+ \frac{1}{n!} \frac{\sin^4 n\pi s \cdot \sin^2 n\pi t}{\sin^8 n\pi s + \sin^4 n\pi t} + \cdots$$

The first term of this series has a discontinuity in s and t at

$$(0, 0),\ (0, 1),\ (1, 0),\ (1, 1);$$

the second term at the points

$$(0, 0),\ (0, \tfrac{1}{2}),\ (0, 1),\ (\tfrac{1}{2}, 0),\ (\tfrac{1}{2}, \tfrac{1}{2}),\ (\tfrac{1}{2}, 1),\ (1, 0),\ (1, \tfrac{1}{2}),\ (1, 1).$$

In the general term we have such a singularity at points for which s and t have any combination of the following values:

$$s = 0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \cdots,\ \frac{n-1}{n},\ 1,$$

$$t = 0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \cdots,\ \frac{n-1}{n},\ 1.$$

It follows that any point whose coördinates (s, t) are both rational numbers is a point of discontinuity of some term in the series defining $F(s, t)$. It will be observed that none of the terms of the series given in (2) are ever negative, whatever values may be assigned to s and t. The points of discontinuity of any term are likewise points of discontinuity of all subsequent terms but not necessarily of previous terms. Moreover, the discontinuities at a given point can not be combined so as to cancel each other. For, if a point first appears as a discontinuity, say in the kth term, the sum of the discontinuities of subsequent terms at that point can not equal in amount the discontinuity of the kth term. For example, consider the discontinuity which appears at the rational point $(x = \frac{1}{2}, y = \frac{1}{3})$. This point will appear as a discontinuity for the first time in the third term. The amount of the discontinuity in this term is $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2} \cdot \frac{1}{6}$. The amount of the disconinuity of subsequent terms at this point can not exceed

$$\frac{1}{4!}\cdot\frac{1}{2}+\frac{1}{5!}\cdot\frac{1}{2}+\cdots<\frac{1}{2}\cdot\frac{1}{4!}\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots\right)=\frac{1}{2}\cdot\frac{1}{4!}\cdot\frac{5}{4}<\frac{1}{2}\cdot\frac{1}{6}.$$

Consequently, all points which appear as points of discontinuity of any term are also points of discontinuity of $F(s, t)$, and hence any point whose coördinates are both rational numbers is such a point. A function must be continuous in the two variables together in order to be totally differentiable. It follows then that $F(s, t)$ as defined is not totally differentiable at any point where the two coördinates are both rational numbers. Such a set of points is everywhere dense in the given region.

We shall now show that the points at which $F(s, t)$ is totally differentiable are also everywhere dense. To do this, we proceed as follows. As we have seen, f_x' is continuous in x in the closed interval $0 \leqq x \leqq 1$;

hence for any constant value of y, say $y = y_0$, the numerical values of $f_x'(x, y_0)$ have a finite upper bound. This upper bound, which we shall denote by M_0, may change with the choice of y_0, but the essential thing is that for each value of y_0 it is a constant. We have then, since $\sin n\pi s$ and $\cos n\pi s$ are numerically less than one,

$$\left|\frac{\partial\varphi_n}{\partial s}\right| = \left|\frac{\partial f}{\partial x}\cdot\frac{dx}{ds}\right| \leqq 2\pi n M_0 \mid \sin n\pi s \cdot \cos n\pi s \mid \leqq 2\pi n \cdot M_0.$$

Hence we have for a constant value of t

$$\sum_{n=1}^{\infty}\left|\frac{1}{n!}\frac{\partial\varphi_n}{\partial s}\right| \leqq \sum_{n=1}^{\infty}\frac{2\pi M_0}{(n-1)!}.$$

As this last series converges, it follows that the series $\Sigma(1/n!)/(\partial\varphi_n/\partial s)$ converges uniformly, and we may write

$$\frac{\partial F}{\partial s} = \sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial\varphi_n}{\partial s}.$$

Moreover, since $\partial\varphi_n/\partial s$ is continuous in s, it follows that $\partial F/\partial s$ is defined by a uniformly convergent series of continuous functions and is therefore itself continuous in s.

By a similar method it may be shown that $\partial F/\partial s$ is continuous in t; and since this derivative is also continuous in s, it follows that there exists a set of points everywhere dense where it is continuous in s and t together.

We shall now consider the existence of the partial derivative $\partial F/\partial t$. From *Ex.* 1, it follows that f_y' is continuous in y in the closed interval $0 \leqq y \leqq 1$. Then, for any constant value of x it is bounded, and hence we may write

$$\left|\frac{\partial\varphi_n}{\partial t}\right| = \left|\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right| \leqq 2n\pi M_1 \mid \sin n\pi t \cdot \cos n\pi t \mid \leqq 2n\pi M_1,$$

where M_1 is a constant for each previously selected value of x. Consequently, we have

$$\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial\varphi_n}{\partial t} \leqq \sum_{n=1}^{\infty}\frac{2\pi M_1}{(n-1)!}.$$

Since this last series converges, it follows that $\Sigma(1/n!)(\partial\varphi_n/\partial t)$ converges uniformly, and hence we may write

$$\frac{\partial F}{\partial t} = \sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial\varphi_n}{\partial t}.$$

Therefore, the partial derivatives $\partial F/\partial s$, $\partial F/\partial t$ both exist at every point in the given region and $\partial F/\partial s$ is continuous in s and t together at a set of points everywhere dense. It follows that $F(s, t)$ is totally differentiable at each of these points.

We have then a function which is totally differentiable at a set of points everywhere dense, and at the same time there exists a set of points likewise everywhere dense at which this function is not totally differentiable, in spite of the fact that both of the first partial derivatives exist at every point and one of them is everywhere continuous in each variable separately. Geometrically, the corresponding surface has a tangent plane at a set of points everywhere dense and there exists simultaneously another set of points at which there is no tangent plane. It can be shown, moreover, that the points at which the tangent planes exist, that is, the points at which the given function is totally differentiable, must form a set having the cardinal number of the continuum. It will be observed that $\partial F/\partial t$ is not continuous with respect to s. By the method used in discussing the continuity of $F(s, t)$, it can be shown that $\partial F/\partial t$ is discontinuous in s at a set of points everywhere dense, namely at the points where s and t are rational numbers.

It has already been pointed out that the mere existence of the partial derivatives f_x', f_y' at a given point is not a sufficient condition for the total differentiability of $f(x, y)$ at that point. The question naturally arises whether it is possible to find a function having these partial derivatives at all points of a given region and yet not be totally differentiable at any point of that region. It can be readily shown that this cannot be the case. For it is known* that if f_x', f_y' exist, they can be at most pointwise discontinuous in the two variables x and y. The points of continuity in x and y together are then everywhere dense and these points are points of total differentiability of the given functions.

If we assume, in addition to the existence of the partial derivatives f_x', f_y', the continuity of these derivatives with respect to x and with respect to y, then we have the following theorem.

THEOREM. *Given a function $f(x, y)$ whose partial derivatives f_x', f_y' are continuous in x and in y and bounded as to x and y taken together in a closed region R. Then $f(x, y)$ is totally differentiable at all points of R.*

From the existence of the partial derivatives f_x', f_y' it follows that $f(x, y)$ is continuous in x and in y. Let (x_0, y_0) be any point in the given region R. Since $f(x, y)$ is continuous with respect to y, we have for each value of x in any closed interval lying wholly in R

$$\mathop{\mathrm{L}}_{y=y_0} f(x, y) = f(x, y_0).$$

Moreover, f_x' is continuous in x in any such interval for $y \neq y_0$. By

* See Baire, Annali di Mat., Series III, vol. 3 (1899), p. 108.

hypothesis, f_x' is also bounded, when considered as a function of the two variables (x, y) together. It follows that for any closed interval taken on $y = y_0$, and lying in R the given function $f(x, y)$ converges uniformly* to the function $f(x, y_0)$. Consequently we have†

$$\mathop{\mathrm{L}}_{\substack{x=x_0\\ y=y_0}} f(x, y) = f(x_0, y_0);$$

or what is the same thing, for an arbitrarily small positive number η, there exists a $\lambda > 0$ such that

$$(3) \qquad |f(x, y) - f(x_0, y_0)| < \frac{\eta}{2}, \qquad |x - x_0| < \lambda, \qquad |y - y_0| < \lambda.$$

Hence, for a given value of η, however small it may be chosen, $\lambda(x, y)$ is defined for each point (x, y) in R. The function $f(x, y)$ is therefore continuous in x and y together in the closed region R, and hence it is uniformly continuous in R and $\lambda(x, y)$ has a lower limit λ_0 greater than zero. There exists then about each point of R as a center a square S_0 whose sides are of length $2\lambda_0$ such that the oscillation of $f(x, y)$ within the square is less than η. We may regard the point (x_0, y_0) the center of such a square.

Since the partial derivative f_x' exists at the point (x_0, y_0), we have for all values of $x_0 + \Delta x$ within an interval $[x_0 - \delta_1(x_0, y_0), x_0 + \delta_1(x_0, y_0)]$

$$(4) \qquad \left|\frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} - f_x'(x_0, y_0)\right| < \eta.$$

It follows that

$$(5) \qquad |f(x_0 + \Delta x, y_0) - f(x_0, y_0) - \Delta x f_x'(x_0, y_0)| < \eta\,|\Delta x|.$$

Since the oscillation of $f(x, y)$ in S_0 is less than η, we have for $|\Delta x| < \lambda_0$, $|\Delta y| < \lambda_0$

$$(6) \qquad |f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)| < \eta,$$

$$(7) \qquad |f(x_0, y_0) - f(x_0, y_0 + \Delta y)| < \eta.$$

Combining (5), (6), and (7), we get

* If $f(x, y)$ is continuous in y for each value of x in an interval $\alpha \leqq x \leqq \beta$, and if f_x' exists and

$$|f_x'(x, y)| \leqq G, \qquad \alpha \leqq x \leqq \beta, \qquad y_0 - \delta \leqq y \leqq y_0 + \delta,$$

where G is a finite number, then $f(x, y)$ converges uniformly to the function $f(x, y_0)$. Cf. Townsend, *Begriff u. Anwendung des Doppellimes* (Göttingen Dissertation), p. 34.

† The necessary and sufficient condition that $f(x, y)$ shall converge uniformly to the boundary function $f(x, y_0)$ in the interval (α, β) is that the double simultaneous limit

$$\mathop{\mathrm{L}}_{\substack{x \doteq x_0\\ y \doteq y_0}} f(x, y) = f(x_0, y_0)$$

for each point in the closed interval $\alpha \leqq x \leqq \beta$. See Townsend, *Ibid.*, p. 39.

(8) $|f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) - \Delta x f_x'(x_0, y_0)| < \eta(2 + |\Delta x|).$

From the existence of the partial derivative $f_y'(x_0, y_0)$, we have for some interval $[y_0 - \delta_2(x_0, y_0), y_0 + \delta_2(x_0, y_0)]$

$$\left|\frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} - f_y'(x_0, y_0)\right| < \eta,$$

whence

(9) $$|f(x_0, y_0 + \Delta y) - f(x_0, y_0) - \Delta y f_y'(x_0, y_0)| < \eta\,|\Delta y|\,.$$

By adding and subtracting $f(x_0, y_0 + \Delta y)$ to the numerator of the first member of the following equation, we have the identity

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)}$$
$$= \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta(x, y)} + \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)}.$$

Transposing all the terms to the first member of the equation, we have upon making use of the relations given in (8) and (9)

$$\left|\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - \Delta x f_x'(x_0, y_0) - \Delta y f_y'(x_0, y_0)}{\Delta(x, y)}\right|$$
$$< \eta(2 + |\Delta x| + |\Delta y|),$$

which holds for all values of $(\Delta x, \Delta y)$ numerically less than δ_0 where δ_0 denotes the smallest of the three numbers λ_0, δ_1, δ_2. The second member of this inequality is arbitrarily small, since η is an arbitrarily small number. Hence the limit given in (7) exists, and the given function is totally differentiable at (x_0, y_0).

That the foregoing theorem gives a sufficient but not a necessary condition for total differentiability is at once evident from the following illustrative example.

Ex. 3. Given $f(x, y) = (x^2 + y^2) \sin 1/(x + y)$ for $x \neq 0$, $y \neq 0$ and $x \neq -y$, and let $f(x, y) = 0$ for $(x = 0, y = 0)$ and for $x = -y$.

This function has at (0, 0) the partial derivatives $f_x'(0, 0)$, $f_y'(0, 0)$; for, we have for $x = 0$, $y = 0$,

$$f_x'(0, 0) = \mathop{\mathrm{L}}_{\Delta x = 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \mathop{\mathrm{L}}_{\Delta x = 0} \Delta x \sin \frac{1}{\Delta x} = 0,$$

$$f_y'(0, 0) = \mathop{\mathrm{L}}_{\Delta y = 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \mathop{\mathrm{L}}_{\Delta y = 0} \Delta y \sin \frac{1}{\Delta y} = 0.$$

For $x = 0$, $y \neq 0$, we obtain $f_x'(0, y) = -\cos 1/y$. Finally, for $x \neq 0$, $y \neq 0$, we have $f_y'(x, 0) = -\cos 1/x$. It follows that at the origin f_x' is discontinuous in y, and f_y' is discontinuous in x. However, the given function is totally differentiable at (0, 0); for, we have

$$\underset{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}}{\mathrm{L}} \left[\frac{f(\Delta x, \Delta y) - f(0, 0) - \Delta x f_x'(0, 0) - \Delta y f_y'(0, 0)}{\Delta(x, y)} \right]$$

$$= \underset{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}}{\mathrm{L}} \left[\sqrt{(\Delta x)^2 + (\Delta y)^2} \sin \frac{1}{\Delta x + \Delta y} \right] = 0.$$